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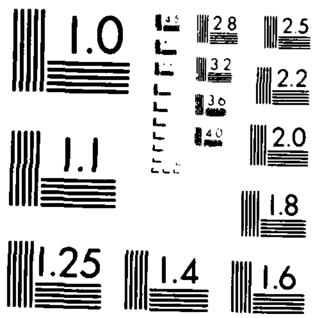

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A CHARACTERIZATION OF AN ELEMENT OF BEST SIMULTANEOUS APPROXIMATION

R. W. Owens

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

Deutsch [4] has suggested that some problems of best simultaneous approximation might profitably be viewed as problems of best approximation in an appropriate product space. A few authors have touched upon this approach; none, however, have pursued it consistently or developed a complete problem along such a line, even in the simplest of cases. In this paper, we show that Deutsch's suggestion can easily be carried out using known results from approximation theory to establish existence, uniqueness, and characterization results. An algorithm guaranteed to converge strongly to the element of best simultaneous approximation under certain circumstances is also proposed.

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* Department of Mathematics, Lewis and Clark College, Portland, OR 97219, USA.

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SIGNIFICANCE AND EXPLANATION

A basic problem of best simultaneous approximation is the following. Given a set S, two (or more) points not in S, and (possibly different) measures of the distances from the points to the set, find the element of S which is, in some sense, simultaneously closest to the given points not in S. Deutsch [4] has suggested that some problems of best simultaneous approximation might profitably be viewed as problems of best approximation in an appropriate product space. A few authors have touched upon this approach; none, however, have pursued it consistently or developed a complete problem along such a line, even in the simplest of cases. In this paper, we show that Deutsch's suggestion can easily be carried out using known results from approximation theory to establish existence, uniqueness, and characterization results. An algorithm guaranteed to converge strongly to the element of best simultaneous approximation under certain circumstances is also proposed

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R. W. Owens*

1. INTRODUCTION

Let X be a Banach space with norm $\|\cdot\|$, M a subset of X, and $x_1, x_2 \in X\backslash M$. Let $\|\cdot\|_\alpha$ be a norm on R^2. The best simultaneous approximation (BSA) problem to be considered, referred to as problem (P), is:

(P) Find $\bar{m} \in M$ minimizing $\{\|(\|x_1-m\|, \|x_2-m\|)\|_\alpha \mid m \in M\}$.

Without further conditions upon X, M, $\|\cdot\|$, and $\|\cdot\|_\alpha$, a solution of (P) may either not exist or not be unique. Existence and uniqueness results for many choices of X, M, $\|\cdot\|$, and $\|\cdot\|_\alpha$ are known; see Sahney and Singh [16] for a recent survey of many such results. Throughout this paper, X will be a strictly convex and reflexive Banach space, M a closed subspace of X, and the norm $\|\cdot\|_\alpha$ on R^2 will be strictly convex and strictly monotone increasing in each coordinate on $\{(x,y) \mid x,y \geq 0\}$.

Characterizations of an element of BSA come in several different forms; again [16] gives a detailed overview of these results. Several authors, e.g. Diaz and McLaughlin [5, 6], Dunham, [7], Holland et.al. [10,11], Ling [12], and Milman [14], have reduced the problem of BSA to a simpler problem of best approximation of a single element, often a weighted mean of the points to be approximated; others, e.g. Brosowski [2], Dunham [7], and Subrahmanya [21], employ alternation systems; while still

* Department of Mathematics, Lewis and Clark College, Portland, OR 97219, USA.

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others, e.g. Goel et.al. [8], Milman [14], and Phillips et.al. [15], give characterizations in terms of one or more linear functionals for which certain conditions are met. It is this latter route along which this paper proceeds and upon which the proposed algorithm is based.

In Section 2, we specify notation, recast the BSA problem as a problem of best approximation in a product space, and establish the existence and uniqueness of the solution. In Section 3, we introduce a dual problem in terms of which our characterization result is given. An algorithm and a guarantee of stong convergence when M has finite co-dimension is presented in Section 4. Finally, in Section 5, generalizations or extensions of the theory are established.

2. BASIC NOTIONS

Let $Z = X \oplus X = \{(x,y) | x,y \in X\}$, $N = \{(m,m) | m \in M\}$, and $b = (x_1, x_2)$. Norm Z with $|z| = \|(\|x\|, \|y\|)\|_\alpha$, where $z = (x,y) \in Z$. The strictly monotone increasing assumption on $\|\cdot\|_\alpha$ ensures that $|\cdot|$ satisfies the triangle inequality; the other conditions needed for $|\cdot|$ to be a norm on Z are clearly satisfied. Note also that $|\cdot|$ is strictly convex.

Thus Z is a strictly convex and reflexive Banach space with norm $|\cdot|$, N is a closed subspace of Z, and $b \in Z \backslash N$. Problem (P) can be rephrased as a simple best approximation problem:

(P) Find $\bar{n} \in N$ minimizing $\{|b-n| \,|\, n \in N\}$.

Clearly the two formulations of problem (P) are equivalent so throughout this paper no distinction between them will be made.

It is known [17] that problem (P) has a unique solution.

For a normed linear space Y, denote by Y^* the dual space of Y, and for $f \in Y^*$, let

$$\|f\| = \sup\{|f(y)| \,|\, y \in Y, \|y\| = 1\}.$$

Let $S = \{y \in Y | \|y\| = 1\}$ and $S^* = \{f \in Y^* | \|f\| = 1\}$. If $A \subset Y$, then $A^\perp = \{f \in Y^* | f(a) = 0 \;\; \forall a \in A\}$, and similarly if $A \subset Y^*$, then $A^\perp = \{y \in Y | f(y) = 0 \;\; \forall f \in A\}$.

2.1 Defintion. If $f \in Y^* \backslash \{0\}$, then $y \in S$ is called a dual vector for f if $f(y) = \|f\|$.

As is well known [22]:

2.2 Theorem. If Y is reflexive and $f \in Y^* \backslash \{0\}$, then there is at least one dual vector for f. Moreover, if Y is strictly convex, then the dual vector is unique.

Under the assumptions on X given above, each $f \in Z^* \backslash \{0\}$ has a unique dual vector which will be denoted by f^*.

3. DUAL PROBLEM

Given problem (P) we can associate with it a dual problem (D).

(D) Find $\bar{f} \in N^{\perp} \cap S^*$ maximizing $\{f(b) \mid f \in N^{\perp} \cap S^*\}$.

It is known [17] that problem (D) has a soltuion, and by an application of the Hahn-Banach Theorem, see e.g. [3], we have that

$$\inf_{m \in M} \|(\|x_1 - m\|, \|x_2 - m\|)\|_{\alpha} = \sup_{f \in N^{\perp} \cap S^*} f(b). \tag{3.1}$$

The specific characterization result that we desire is a special case of the following lemma. The lemma is certainly known, but we have been unable to locate it in the literature. For completeness, a sketch of the proof is included.

3.1 Lemma. Let Y be a strictly convex Banach space and K a subspace of Y. For $b \in Y$ we denote by $(\underline{P})$ and $(\underline{D})$ the follwoing primal and dual problems.

$(\underline{P})$ Find $y_0 \in K$ such that $\|b - y_0\| = \inf_{y \in K} \|b - y\|$.

$(\underline{D})$ Find $\bar{f} \in K^{\perp} \cap S^*$ such that $\bar{f}(b) = \sup_{f \in K^{\perp} \cap S^*} f(b)$.

Suppose also that $\inf_{y \in K} \|b - y\| = \rho > 0$. Then $(\underline{P})$ has a solution y_0 if and only if the solution $\bar{f}$ of $(\underline{D})$ has a dual vector $\bar{f}^*$, in which case $y_0 = b - \bar{f}(b)\bar{f}^*$.

Proof sketch: We first observe that whether $(\underline{P})$ has a solution or not, $(\underline{D})$ does, see e.g. Luenberger [13], so that the existence of $\bar{f}$ is assured. Moreover, as in the derivation of (3.1), we have that $\bar{f}(b) = \inf_{y \in K} \|b - y\| = \rho > 0$. It is also well known, see [1] or [18], that due to the strict convexity of Y, $\bar{f}$ is unique.

The calculation

$$\|b - y_0\| = \|\bar{f}(b)\bar{f}^*\| = \bar{f}(b)\|\bar{f}^*\| = \bar{f}(b) = \rho$$

shows that if $\bar{f}^*$ exists, then $y_0 = b - \bar{f}(b)\bar{f}^*$ solves $(\underline{P})$.

Conversely, let y_0 be the solution of ($\underline{P}$). We show that $(b-y_0)/\bar{f}(b)$ is the dual vector for $\bar{f}$. Since $\bar{f}(b) > 0$, $\bar{f} \neq 0$. Also, $\bar{f}\big((b-y_0)/\bar{f}(b)\big) = 1 = \|\bar{f}\|$ since $y_0 \in K$ and $\bar{f} \in K^{\perp} \cap S^*$. Thus by Definition 2.1, $\bar{f}^* = (b-y_0)/\bar{f}(b)$, and solving for y_0 we obtain the desired formula for y_0.

The hypotheses under which our BSA problem is being considered ensure the applicability of Lemma 3.1. We have taken Z to be strictly convex. Moreover, since Z was chosen to be reflexive and N a closed subspace of Z, N is proximinal so that for any $b \in Z$ the primal problem has a unique solution. Thus the dual vector for the solution of problem (D) will exist and the solution of problem (P) can be formulated in such terms.

Rephrasing the lemma in the specific terms of our BSA problem, we have the following characterization of the solution of our original problem.

<u>3.2 Theorem.</u> Let $\bar{f}$ be any solution of the dual problem (D). Then (i) there exists a unique $\bar{m} \in M$ such that, with $\bar{n} = (\bar{m},\bar{m})$, $b - \bar{f}(b)\bar{f}^* = \bar{n}$, and (ii) $\bar{m}$ is the solution of (P).

4. ALGORITHM

The problem formulation and characterization result of the previous section follow an approach used by Sreedharan [19] to determine solutions of overdetermined systems of linear equations. The approach and algorithms introduced by Sreedharan were later generalized by Anton and Duris [1]. Since only minor modifications of the algorithm given by Anton and Duris [1] for computing a solution of the dual problem (D) are needed, we present only the algorithm and the main convergence result. The reader is referred to [1] for the proofs; the changes needed are minor and straightforward.

4.1 Definition. Let W be a finite dimensional subspace of Z^* and let $B = \{g_1, g_2, \dots, g_k\}$ be a basis for W. The projection of Z onto W relative to B is defined to be $F: Z \to W$ by $F(z) = \sum_{i=1}^{k} g_i(z) g_i$.

Note that $F(z) = 0$ if and only if $z \in W^{\perp}$.

Motivation for an algorithm to compute a solution of problem (D) comes from the following

4.2 Lemma. Let $f \in N^{\perp} \cap S^*$ with $f(b) > 0$, and let $F: Z \to (N \cup \{b\})^{\perp}$ be the projection of Z onto $(N \cup \{b\})^{\perp}$ relative to any basis B of $(N \cup \{b\})^{\perp}$. Then f solves problem (D) if and only if $F(f^*) = 0$.

Proof: See [1].

Led by this characterization of a solution of the dual problem (D), the following algorithm is proposed for the solution of problem (D) when the subspace M, equivalently N, has finite codimension.

4.3 Algorithm. Step (1): Select a basis $B = \{g_1, g_2, \dots g_k\}$ for $W = (N \cup \{b\})^{\perp}$, and let F be the projection of Z onto W relative to B. Set $i = 0$.

Step (2): Choose $f_0 \in N^{\perp} \cap S^*$ with $f_0(b) > 0$.

Step (3): Compute $h_i = F(f_i^*)$.

Step (4): If $h_i = 0$, then f_i solves problem (D) by Lemma 4.2; if $h_i \neq 0$ then go to step (5).

Step (5): Determine α such that

$$\|f_i - \alpha h_i\| \leq \|f_i - \lambda h_i\| \;\forall \lambda \in \mathbb{R}.$$

Step (6): Set $f_{i+1} = \dfrac{f_i - \alpha h_i}{\|f_i - \alpha h_i\|}$, increase i by 1, and go to step (3).

4.4 Theorem. Let Z and Z^* be uniformly convex and let N have finite codimension. Then Algorithm 4.3 either produces a solution of problem (D) in a finite number of steps or generates a sequence f_i which converges strongly to the solution of problem (D).

The proof of the convergence of Algorithm 4.3 can be found in [1]. Sreedharan [18, 19, 20] has developed several variations of the algorithm. In addition, he discusses ℓ_p and weighted ℓ_p cases including the computation of dual vectors needed both in step (3) of the algorithm and in finding the solution of problem (P) once the solution of the dual problem is known.

5. CONCLUDING REMARK

Consider the following more general problem of simultaneous approximation of n elements:

$$\text{Find } \bar{m} \in M \text{ minimizing}$$

$$\left\{ \|\left(\|x_1 - m\|_{\alpha_1}, \|x_2 - m\|_{\alpha_2}, \ldots, \|x_n - m\|_{\alpha_n}\right)\|_{\alpha} \middle| m \in M \right\},$$

where the n norms $\|\cdot\|_{\alpha_i}$, $i=1,2,\ldots,n$, are strictly convex and the norm $\|\cdot\|_{\alpha}$ on R^n is strictly convex and strictly monotone increasing in each of its n components on $\{(y_1,\ldots,y_n) | y_i > 0\}$. Each of the results of this paper extends easily to encompass such a problem.

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